

# Gauge Fixing and Observables in General Relativity

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The conventional group of four-dimensional diffeomorphisms is not realizable as a canonical transformation group in phase space. Yet there is a larger field-dependent symmetry transformation group which does faithfully reproduce 4-D diffeomorphism symmetries. Some properties of this group were first explored by Bergmann and Komar. More recently the group has been analyzed from the perspective of projectability under the Legendre map. Time translation is not a realizable symmetry, and is therefore distinct from diffeomorphism-induced symmetries. This issue is explored further in this paper. It is shown that time is not "frozen". Indeed, time-like diffeomorphism invariants must be time-dependent. Intrinsic coordinates of the type proposed by Bergmann and Komar are used to construct invariants. Lapse and shift variables are retained as canonical variables in this approach, and therefore will be subject to quantum fluctuations in an eventual quantum theory. Concepts and constructions are illustrated using the relativistic classical and quantum free particle. In this example concrete time-dependent invariants are displayed and fluctuation in proper time is manifest.

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## I. HOMAGE AND INTRODUCTION

What better way to celebrate the birthday of our dear friend and teacher Balachandran than to continue to reflect and debate together, with the same intellectual rigor, impassioned commitment and irreverent playfulness most of us recall from the decades-old tradition of Physics Building Room 316. And what a joy it has been to gather together generations of physicists who were formed by this process. Thank you Bal for your inspirational example. Alas, though your explorations in quantum geometry have long stimulated my own thinking about the nature of time, I'm afraid I never did learn master your skill in using time wisely!

My collaborators J.M.Pons, L. C. Shepley, and I have recently elucidated the nature of four-dimensional diffeomorphism symmetries in canonical phase space formulations of general relativity[1, 2, 3, 4]. It turns out that a global constant translation in time (time evolution) is not realizable as a symmetry transformation. Thus symmetries and time evolution are mathematically distinct. The incorrect identification of time evolution as a symmetry has led to the so called "problem of time", the assertion that diffeomorphism invariants must be independent of time. Various routes to this mistaken conclusion are analyzed by J. M. Pons and myself in a forthcoming paper. [5] I will review here the general framework, with special emphasis on the retention of lapse and shift variables as canonical variables. They play an essential role in the diffeomorphism-induced symmetry transformation group, and in addition there is good reason for promoting them to quantum variables subject to fluctuation. This property will be displayed explicitly in a free relativistic particle example. Diffeomorphism invariants will be constructed through the use of intrinsic coordinates, coordinates whose values are fixed by the values of physical fields. These invariants in general relativity are necessarily time dependent.

## II. LEGENDRE PROJECTABILITY OF DIFFEOMORPHISM SYMMETRIES

All generally covariant theories have the property that variations of physical variables generated by diffeomorphisms which alter the time are not projectable under the Legendre transformation to phase space. I will give the generic explanation and then illustrate with the relativistic free particle. Suppose we have a system described by a Lagrangian  $L(q, \dot{q})$ , with configuration variables  $q^i$  and velocities  $\dot{q}^i$ . Then due to the general covariance of the model the Legendre matrix is singular:  $\det \frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j} = 0$ . Thus functions  $f(q, \dot{q})$  which vary along the null directions of this matrix are not projectable under the Legendre map  $p_i = \frac{\partial L}{\partial \dot{q}^i}$ , as I now show. Let  $\gamma^i$  represent the components of a null vector, i.e.,  $\frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j} \gamma^i = 0$ . More precisely I mean by "projectable" that  $f$  is the pullback of a function  $F(q, p)$  in phase space. If

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this is the case it follows that

$$\gamma^i \frac{\partial f(q, \dot{q})}{\partial \dot{q}^i} = \frac{\partial F(q, p(q, \dot{q}))}{\partial p_k} \gamma^i \frac{\partial^2 L}{\partial \dot{q}^k \partial \dot{q}^i} = 0. \quad (1)$$

and the assertion is proved. Consider as an example the relativistic free particle described by the Lagrangian  $L = \frac{1}{N} \dot{x}^2 - \frac{N}{2}$ , where  $x^\mu(t)$  is the Minkowski spacetime position, dependent on the arbitrary parameter  $t$ . The variable  $N$  is the lapse which determines the proper time elapsed as a function of  $t$ :  $d\tau = N(t)dt$ . The resulting equations of motion are covariant under arbitrary changes in the parameterization. Consequently, the Legendre matrix possesses a null direction, given in this case by  $\frac{\partial}{\partial N}$ , so projectable functions may not depend on  $\dot{N}$ .

In fact, the time derivatives of the lapse and shift functions are absent in all generally covariant theories, including general relativity. The lapse  $N$  and shift  $N^a$  appear in the 3 + 1 decomposition of the spacetime metric

$$g_{\mu\nu} = \begin{pmatrix} -N^2 + g_{cd}N^cN^d & g_{ac}N^c \\ g_{bc}N^c & g_{ab} \end{pmatrix}. \quad (2)$$

Let us calculate the variations of the metric (the Lie derivative) induced by infinitesimal coordinate transformations of the form  $x'^\mu = x^\mu - \epsilon^\mu(x)$ , where the  $\epsilon^\mu$  are arbitrary infinitesimal functions,

$$\delta g_{\mu\nu} = g_{\mu\nu,\alpha} \epsilon^\alpha + g_{\alpha\nu} \epsilon_{,\mu}^\alpha + g_{\mu\alpha} \epsilon_{,\nu}^\alpha. \quad (3)$$

We note that variations of the lapse and shift do depend on these time derivatives. Therefore all these variations corresponding to non-vanishing  $\epsilon^0$  are not projectable. For example, we find in our free particle example that under an arbitrary infinitesimal reparameterization  $t' = t - \epsilon(t)$  the variation of  $N$  is  $\delta N = \dot{N}\epsilon + N\dot{\epsilon}$ .

Nevertheless there does exist a gauge symmetry group which reproduces on any given solutions of the equations of motion all symmetry transformations engendered by the conventional diffeomorphism group. This trick is accomplished by broadening the group to include a compulsory dependence on the lapse and shift. It is straightforward to prove that the unique form of  $\epsilon$  which produces projectable variations of the lapse and shift is [1],

$$\epsilon^\mu(x, N, N^a) = \delta_a^\mu \xi^a(x) + n^\mu(x) \xi^0(x), \quad (4)$$

where  $n^\mu$  is the orthogonal vector to the time foliation of spacetime and is given by

$$n^\mu = (N^{-1}, -N^{-1}N^a). \quad (5)$$

The  $\xi^\mu$  are arbitrary infinitesimal functions of the coordinates. This familiar decomposition was first introduced by Dirac, though the group theoretical explanation was not known until decades later. In fact this form is not quite correct since, as was first pointed out by Bergmann and Komar [6], the group multiplication rule in general relativity produces an unavoidable non-local dependence of  $\xi^0$  on the spatial metric. Returning to our example we note that under  $t' = t - N^{-1}\xi(t)$  we find

$$\delta N = \dot{N}N^{-1}\xi + N \frac{d}{dt} (N^{-1}\xi) = \dot{\xi}. \quad (6)$$

### III. SYMMETRY GENERATORS AND THE HAMILTONIAN

The general structure of the generators of diffeomorphism-induced symmetries, in which the lapse and shift are retained as canonical variables, is

$$G[\xi] = \int d^3x \left( \dot{\xi}^\mu P_\mu + \xi^\mu \left( \mathcal{H}_\mu + \int \int d^3y d^3z C_{\mu\alpha}^\beta(x, y, z) N^\alpha(y) P_\beta \right) \right). \quad (7)$$

In this expression the  $P_\mu$  are the momenta conjugate to the lapse and shift. They are constrained to vanish. Preservation of these constraints in time leads to the secondary constraints  $\mathcal{H}_\mu \approx 0$ . These constraints form a closed equal-time Poisson Bracket algebra with structure functions  $C_{\mu\alpha}^\beta$ :

$$\{\mathcal{H}_\mu(x^0, \vec{x}), \mathcal{H}_\nu(x^0, \vec{y})\}_{PB} = \int d^3z C_{\mu\nu}^\beta(x, y, z) \mathcal{H}_\beta. \quad (8)$$

Some  $C$ 's depend on the spatially differentiated three-metric and this is the origin of the non-local group dependence on the three-metric noted above. The generator in our free particle example is

$$G[\xi] = \dot{\xi}P + \xi \frac{1}{2}(p^2 + 1). \quad (9)$$

These generators are distinct from the Hamiltonian which is of the form

$$H = \int d^3x (N^\mu \mathcal{H}_\mu + \lambda^\mu P_\mu). \quad (10)$$

In this expression  $\lambda(x)$  are monotonically increasing functions of  $x^0$  but otherwise arbitrary. Note that in contrast with the symmetry generator  $G[\xi]$  the secondary constraints in  $H$  are multiplied by the canonical variables  $N^\mu$ , and not by arbitrary functions. Note also that a time-dependent choice of  $\lambda$  implies a time-dependent Hamiltonian. The Hamiltonian for the free particle is  $H = \frac{N}{2}(p^2 + 1) + \lambda P$ .

#### IV. FINITE TIME EVOLUTION AND SYMMETRY TRANSFORMATIONS

The finite time evolution operator which evolves data from time zero to  $t$  is

$$\begin{aligned} \hat{U}(t, 0) &= \mathcal{T} \exp \left( \int_0^t dt' \{ \quad, H(t') \}_{PB} \right) \\ &= 1 + \int_0^t dt' \{ \quad, H(t') \}_{PB} \\ &\quad + \int_0^t dt_1 \int_0^{t_1} dt_2 \{ \{ \quad, H(t_1) \}, H(t_2) \}_{PB} + \dots \end{aligned} \quad (11)$$

In the first line  $\mathcal{T}$  denotes time ordering. It is straightforward to compute the time evolution of variables in our free particle model. We let the variables with no time argument denote their initial values (and hence their initial phase space coordinate). In addition to  $p_\mu(t) = p_\mu$  we find

$$N(t) = \hat{U}(t, 0)N = N + \int_0^t dt_1 \lambda(t_1), \quad (12)$$

and

$$x^\mu(t) = \hat{U}(t, 0)x^\mu = x^\mu + \int_0^t dt_1 N(t_1)p^\mu. \quad (13)$$

The finite symmetry operator  $\hat{S}(s)$  can now be applied to solutions to create a one-parameter family of gauge-transformed solutions associated with the finite group descriptors  $\xi$ . If  $\xi$  is chosen to be independent of the group parameter  $s$  we have

$$\hat{S}(s) = \exp(s \{ \quad, G[\xi] \}_{PB}). \quad (14)$$

(An  $s$  dependence in  $\xi$  would imply an  $s$  ordering in this expression. It turns out that such an  $s$  dependence can be exploited to simplify the resulting general coordinate transformation [7].) In our free particle example  $\hat{S}(s) = \frac{\xi}{2}(p^2 + 1) + \dot{\xi}P$ , and it generates the following one-parameter family of gauge transformed solutions:

$$N_s(t) := \hat{S}(s)N(t) = N(t) + s\dot{\xi}(t), \quad (15)$$

and

$$x_s^\mu(t) := \hat{S}(s)x^\mu(t) = x^\mu(t) + s\xi(t)p^\mu. \quad (16)$$

It is evident in this example that the symmetry transformation effectively alters the original choice of  $\lambda$  in the Hamiltonian. In general we note we now have at our disposal a symmetry transformation which arbitrarily alters the original choice of time foliation. That is, in spite of having been forced to choose a foliation which appears to destroy the full four-dimensional diffeomorphism symmetry, this symmetry is indeed present after all and can in principle be implemented in canonically quantized models!

## V. GAUGE FIXING AND INTRINSIC COORDINATES

We will construct invariants under the diffeomorphism-induced group through the use of intrinsic coordinates. This is a program first proposed by Einstein himself in reconciling himself with the concept of general covariance. The dilemma he faced and his resolution is nicely reviewed in J. Stachel's discussion of the "hole argument" [8]. More recently Komar and Bergmann proposed the use of Weyl scalars in the definition of intrinsic coordinates. In addition they showed how these scalars can be expressed on solution trajectories in terms of canonical phase space variables [9].

Since by assumption the prescription for passage to intrinsic coordinates is unique, all observers (independent of their initial coordinate choice) will agree on the numerical values of all geometric objects provided their solutions lie on the same diffeomorphism-induced equivalence class of solutions. The operational procedure a user would implement would be to set intrinsic coordinates equal to a prescribed set of functions of scalars formed from the physical fields. Then a coordinate transformation of physical variables would be undertaken from the original arbitrarily chosen coordinate system to the intrinsic coordinate system.

This procedure can equivalently be viewed as a gauge fixing. From this point of view we could for instance, in our relativistic free particle example, set the parameter time coordinate equal to a function of the reparametrization scalar  $x^0(t)$ ,

$$t - f^{-1}(x^0(t)) \approx 0, \quad (17)$$

where  $f$  is a monotonically increasing function. Given an arbitrary initial parameterization this constraint will not generically be fulfilled. But because we can now implement a canonical symmetry transformation we can move this solution along the gauge orbit to the solution which does satisfy the gauge condition constraint. The required finite one-parameter family descriptor will then depend on the original solution. (We will take the parameter value  $s = 1$ ). The resulting gauge transformed solution is by construction a diffeomorphism-induced invariant, i.e., an observable; a variation of the original coordinization (which does not satisfy the gauge constraint) results in no change in the gauge-transformed solution. We will now carry out this procedure in detail.

Using (16) we set the gauge-transformed  $x_{s=1}^0(t)$  equal to  $f(t)$

$$f(t) = x^0(t) + \xi[x](t)p^0. \quad (18)$$

Thus we find that the required solution-dependent descriptor is  $\xi[x](t) = \frac{1}{p^0}(f(t) - x^0(t))$ . Substituting this descriptor into (16) we get the invariants

$$x_{\xi[x]}^a(t) = x^a(t) + \frac{p^a}{p^0}(f(t) - x^0(t)) = x^a + \frac{p^a}{p^0}(f(t) - x^0), \quad (19)$$

and

$$N_{\xi[x]}(t) = N(t) + \dot{\xi}[x](t) = \frac{1}{p^0} \frac{df(t)}{dt}. \quad (20)$$

These functions, in addition to  $x^0(t) = f(t)$  are invariant under diffeomorphism-induced symmetry transformation as we now show explicitly.

Before applying the symmetry generator (9) we need to observe that the time evolved constraint variable  $P(t) = P - \frac{1}{2}(p^2 + 1)t$ . Then the only non-vanishing symmetry variations generated by  $G[\eta]$  for infinitesimal  $\eta$  are

$$\delta x^\mu = \{x^\mu, \dot{\eta}(t)P(t) + \eta(t)\frac{1}{2}(p^2 + 1)\}_{PB} = (\eta(t) - t\dot{\eta}(t))p^\mu, \quad (21)$$

and  $\delta N = \dot{\eta}$ . Therefore since  $x_{\xi[x]}^0(t) = f(t)$  and  $N_{\xi[x]}(t) = \frac{1}{p^0} \frac{df(t)}{dt}$  do not depend on these canonical variables they are trivially invariant. In addition,

$$\delta x_{\xi[x]}^a(t) = \delta x^a - \frac{p^a}{p^0} \delta x^a = 0. \quad (22)$$

We have here an explicit construction of invariants which depend on time. Invariants in general relativity will share this characteristic. In fact, we now reinterpret a theorem due to Torre [10] as a proof that no *time-independent* invariants exist in general relativity since the theory contains no additional symmetry beyond general covariance.

## VI. IMPLICATIONS FOR QUANTUM GRAVITY

The program I have described can in principle be applied in general relativity using Weyl scalars to fix intrinsic coordinates. There is of course a significant practical problem which must be addressed even at the classical level: it will be necessary to find a general temporally monotonically increasing function of Weyl scalars. It is probable that such functions can only be found in coordinate patches which must then be pieced together. Or one may perhaps be forced to abandon the attempt to define time purely gravitationally and one may need to resort to the use of material fields.

On the other hand it should now be clear that quantum time evolution can be given a sensible meaning. We are led to consider an improved Wheeler-DeWitt formalism which takes this evolution into account. A Hamilton-Jacobi approach which retains the lapse and shift is also an attractive candidate for making the transition to quantum mechanics. Regarding the use of lapse and shift, there are two paths we might contemplate. We could perform a group average over the diffeomorphism-induced symmetry group. Or we could pursue the gauge fixing formalism outlined in this paper and solve for the metric variables, including lapse and shift, in terms of the true degrees of freedom of the gravitational field.

Before embarking on the quantum implementation of this latter program in our free particle model it will be instructive to review the advantages the retention of lapse and shift will yield in quantum gravity. On the one hand it will in principle be possible to convert from one choice of intrinsic coordinates to another. Of course, to do so we must find a suitable choice of factor ordering which preserves the gauge symmetry algebra. But perhaps more importantly, in addition to our ability to alter the quantum time foliation, we will introduce quantum fluctuation in the full spacetime metric. Current connection approaches to quantum gravity have found a discretization of spacelike 2-surfaces and volumes [11]. A quantum theory which truly reflects an underlying four dimensional diffeomorphism symmetry (if only with respect to a small deformation of the time foliation) must possess a timelike discretization. Operators representing timelike surfaces can be constructed in the framework of the connection approach, but they require additional physical geometric objects, namely the lapse and shift in addition to the temporal component of the connection. The classical symmetry analysis has been extended to the connection formalism (with arbitrary Immirzi parameter) [4, 12], and work on canonical quantization is in progress.

Let us briefly return to the free particle. We can carry out a conventional quantization in which the one-particle Hilbert space is spanned by a spatial momentum basis  $|\vec{p}\rangle$ . We make the intrinsic time choice  $x^0(t) = f(t)$ . This means in practice that observers have rate-adjusted their clocks in this manner. Now we note from (20) that the quantum lapse operator is given by

$$\hat{N}(t) = \frac{1}{\sqrt{\hat{p}^2 + 1}} \frac{df(t)}{dt}, \quad (23)$$

so the elapsed proper time

$$\Delta\hat{\tau} = \int_{t_1}^{t_2} dt \hat{N}(t) = (f(t_2) - f(t_1)) \frac{1}{\sqrt{\hat{p}^2 + 1}}, \quad (24)$$

is subject to quantum fluctuation!

## VII. CONCLUSIONS

We have observed that canonical classical general relativity is covariant under symmetry transformations which are induced by the full four-dimensional diffeomorphism symmetry group. Misunderstandings about the nature of this group have led some to the mistaken conclusion that diffeomorphism invariants must be constant in time. In fact, quite the opposite is true. In general relativity invariants exist and cannot be constant in time. Such invariants can be constructed through a choice of intrinsic coordinates and we have displayed an example of time-dependent invariants for the relativistic free particle. Similar misconceptions have led to the mistaken conclusion that a choice of time foliation of spacetime leaves only the spatial diffeomorphism group as the remaining symmetry group. This is not the case, and one can in fact implement symmetry transformations which arbitrarily alter the foliation. Finally, there is good physical rationale for retaining the lapse and shift as classical and quantum variables. They must be retained to exploit the full four-dimensional symmetry, and in the quantum model they become subject to a desirable quantum fluctuation.

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